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PARTIALLY EFFICIENCY BALANCED DESIGNS AND THEIR APPLICATIONS

P. D. PURI*

Department of Statistics, Haryana Agricultural University
HISAR, INDIA

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Partially efficiency balanced (PEB) designs were introduced by Puri and Nigam (1976). These designs are available in varying replications and/or varying block sizes. In the present paper we demonstrate their applications in various fields.

1. INTRODUCTION

Block designs are widely used in many fields of research. The most common type being randomized complete block (RB) designs. When the number of treatments is too large to preserve homogeneous conditions within a complete block, either balanced incomplete block (BIB) designs or partially balanced incomplete block (PBIB) designs are used. These designs are restricted to equal number of treatment replications and equal block sizes. This is a serious practical obstacle in many possible experimental circumstances. To meet the requirements of the experimenter in unconventional circumstances some designs with varying replications and varying block sizes are available in literature.

A further inconvenience in the use of known incomplete block designs is that they are not available for every parametric combination. Sometimes, a design is available for v treatments in desired block sizes and desired replications, but not necessarily for v+t or v-t treatments, where t is some positive integer. If the number of treatments is such that no suitable design is available, the experimenter usually discards one or

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* *Current mailing address:* Division of Statistics, Sebha University, PO BOX 18755, Sebha, Libya.

more treatments, or applies a design for a larger number of treatments by repeating some of the treatments more than once. This practice may sometimes be uneconomical or undesirable.

In practice the researchers usually plan their experiments with the objective of estimating and testing certain contrasts (comparisons) with the maximum efficiency. The objective of comparisons changes from experiment to experiment. An experimental plan which is optimal for one type of experiment may not be suitable for another experiment. For instance, whereas a BIB design may be quite useful for variety trials it may be irrelevant for biological assays, factorial experiments and for many genetic experiments.

The strategy should be to select a design which meets the requirements of experimental situations rather than selecting the material to meet the requirements of experimental design without bothering about the objective of a comparison. Before selecting a design, it is essential that the researcher should know the basic properties of the design in advance, particularly about the efficiency factor associated with various basic contrasts and he should select a design having the pattern which provides the estimate of the contrasts of major interest with maximum efficiency in addition to keeping the analysis as simple as possible.

In the present paper, we shall discuss the design patterns suitable for various experimental circumstances. The design satisfying these patterns will have simple analysis in addition to estimating contrasts of major interest with maximum efficiency. However, we shall not discuss the method of their construction here.

2. NOTATIONS AND SOME DEFINITIONS

Consider a block design D(v,b,r,k) with v treatments arranged in b blocks of sizes $k_1, \ldots, k_b$ such that the i-th treatment be replicated r_i times, i = 1,..., v. Let

$$r = (r_1, \ldots, r_v)', \qquad k = (k_1, \ldots, k_b)',$$

$$R = \mathrm{diag}(r_1, \ldots, r_v), \qquad K = \mathrm{diag}(k_1, \ldots, k_b),$$

and let $n = \sum_i^v r_i = \sum_j^b k_j$ denote the number of experimental units.

If $r_i = r$ for all i, the design is called equi-replicated and if $k_j = k$ for all j, it is called equi-block sized (or proper). Let $N (=n_{ij})$ be the vxb incidence matrix, where n_{ij} denotes the number of times the i-th treatment occurs in the j-th block. The design is called binary if n_{ij} takes values 0 or 1 for all i and j, otherwise it is called a non-binary design.

Under the usual linear model the least squares equation for the vector t of treatment parameters takes the form

$$Ct = Q,$$

where $C = R - NK^{-1}N'$ and $Q = T - NK^{-1}B$ with T and B being vectors of treatment and block totals, respectively. The matrix C is singular and its rank is equal to $v - 1$ for a connected design. Here we shall restrict our discussion to connected designs only.

Tocher (1952) has defined the matrix Ω^{-1} as

$$\Omega^{-1} = R - NK^{-1}N' + (1/n)rr' = C + (1/n)rr' . \qquad (1)$$

This matrix is non-singular for a connected design. The estimate of a treatment vector t can be taken as

$$\hat{t} = \Omega Q \qquad (2)$$

and the variance-covariance matrix of $\hat{t}$ is

$$v(\hat{t}) = \Omega \sigma^2 , \qquad (3)$$

where σ^2 is the intrablock variance of a single observation.

The main task of the analysis of a block design is to find the inverse of Ω^{-1}. Calinski (1971) has proposed an iterative procedure of inverting Ω^{-1} and has shown that

$$\Omega = (I + \sum_{h}^{\infty} M_0^h)R^{-1},$$

where

$$M_0 = M - (1/n)1r', \quad M = R^{-1}NK^{-1}N'. \qquad (4)$$

This iterative procedure may be quite useful in the analysis when Ω^{-1} is unstructured. But when Ω^{-1} is structured, then it may be easier to determine the pattern of Ω through direct methods. Pearce (1960) classified designs according to the patterns of Ω^{-1}.

The matrix M_0 plays an important role in the analysis and in determining the properties of a design. The simplification of the general formula depends entirely on the pattern of this matrix which also determines the efficiency factor of the design. Further, its relation to the treatment contrasts is also useful in designing block experiments with specific desirable properties. We shall study some of the properties of this matrix here.

Definition 2.1. A linear function s'T, where T is a vector of treatment totals, is called a treatment contrast if $s'r = 0$. It will subsequently be written as s.

Definition 2.2. A linear function c't of treatment parameters is called a parametric contrast if $c'1 = 0$.

It follows from the paper of Jones (1959) that if for some treatment contrast s'T, s is the right eigenvector of M (or M_0) corresponding to the eigenvalue μ, then the parametric contrast c't is estimated with the efficiency factor $1 - \mu$, where $c = Rs$. The treatment contrast s and the

corresponding parametric contrast $c = Rs$ were termed as basic contrasts by Pearce *et al.*(1974) due to the special role played by them in the analysis of the block design.

Definition 2.3.(Puri and Nigam, 1977) A block design D(v,b,r,k) is said to be partially efficiency balanced with m efficiency classes (PEB(m)) if the complete set of v - 1 mutually orthogonal basic treatment contrasts can be partitioned into m disjoint classes such that the efficiency factor associated with every contrast of the i-th class is $(1 - \mu_i)$, where μ_i, i = 1, ..., m, are eigenvalues of M_0 with multiplicities ρ_i, $\sum_i^m \rho_i = v - 1$.

For a PEB(m) design all ρ_i contrasts s_{ij}, $j = 1,\ldots,\rho_i$, belonging to the i-th class satisfy

$$M_0 s_{ij} = \mu_i s_{ij}, \quad j = 1,\ldots, \rho_i. \tag{5}$$

The parameters of a PEB(m) design can be written as

$$v,\ b,\ r,\ k,\ \mu_i,\ \rho_i,\ L_i,\ i = 1,\ldots, m, \tag{6}$$

where

$$L_i = \sum_j^0 i\ (s'_{ij}Rs_{ij})^{-1}s_{ij}s'_{ij}R, \quad i = 1,\ldots, m. \tag{7}$$

For a PEB(m) design the matrices M and M_0 have the spectral decompositions

$$M = \sum_{i=0}^{m} \mu_i L_i \quad \text{and} \quad M_0 = \sum_{i=1}^{m} \mu_i L_i, \tag{8}$$

respectively, where $\mu_0 = 1$ and $L_0 = (1/n)1r'$. The matrix Ω for a PEB(m) design is

$$\Omega = \left\{I + \sum_{i=1}^{m} [\mu_i/(1 - \mu_i)]L_i\right\}R^{-1} \tag{9}$$

and the sums of squares attributed to treatments is

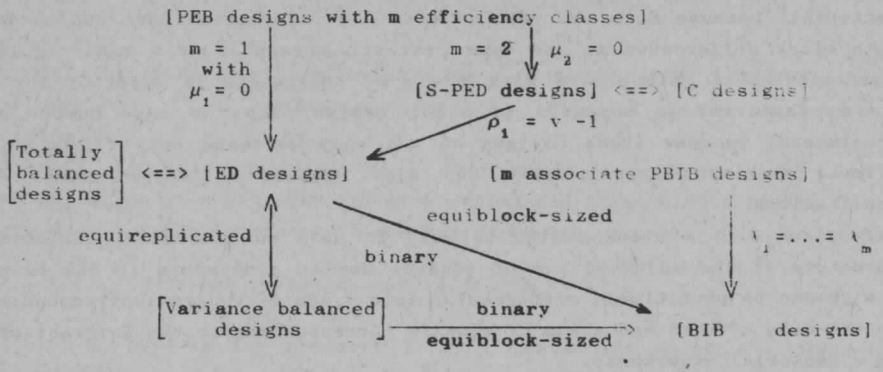
$$\sum_{i=1}^{m} \{1/(1 - \mu_i)\}Q'L_iR^{-1}Q.$$

A particular class of PEB designs, where μ_i takes only two distinct values μ and 0, with multiplicities ρ and $v - \rho - 1$, respectively, is called a simple PEB (S-PEB) design. If $\mu_i = \mu$ for all i, the design is called an efficiency balanced (EB) design (Puri and Nigam, 1975, and Williams, 1975).

We draw the relationship among some block designs in Fig.1. The symbol "A $\xrightarrow[c]{}$ B" means that A implies B under c; "A <==> B" means that A is equivalent to B; "A $\xleftrightarrow[c]{}$ B" means that A and B are equivalent under c.

Remarks. The notion C-designs was introduced by Saha (1976), and a totally balanced design by Caliński (1971). (i) A partially balanced

Fig. 1. The relationship among some block designs



incomplete block (PBIB) design is an equi-replicated, equiblock-sized binary PEB design (cf. Puri and Nigam, 1977). (ii) A supplemented block design, and a (partially) balanced factorial experiment, a linked block design, and a block design having a general balance property of Wilkinson (1970) are a special case of PEB designs. (iii) It is clear from the definition of a PEB design that any connected block design is a special case of a PEB design with at most $v - 1$ efficiency classes, where v is the number of treatments. In this sense, a PEB design is too much general.

We shall now present the applications of a PEB design in various fields. The properties and methods of their construction are available in Puri and Nigam (1977, 1978, 1983), Pal (1980), Nigam and Puri (1982), Ceranka (1983), Puri (1984), Puri and Kageyama (1984, 1985), Kageyama and Puri (1985a, 1985b). Since a design uniquely determines its incidence matrix and conversely, the design and its incidence matrix will be denoted by the same symbol.

3. PEB DESIGN AS FACTORIAL DESIGNS

In factorial experiments, we simultaneously study the effect of a number of factors, each with several levels. Consider a factorial experiment with h factors $F_1, \ldots, F_h$, the j-th factor being experimented with m_j levels, $j = 1, \ldots, h$. Any incomplete block design with $v = m_1 m_2 \ldots m_h$ treatments can be utilized to conduct a $m_1 \times m_2 \times \ldots \times m_h$ factorial experiment by replacing the i-th treatment by the i-th treatment combination.

In factorial experiments the interest shifts from elementary contrasts to the main effects and lower order interactions and thus the conventional incomplete block designs, such as BIB and PBIB, designs may

or may not be suitable as factorial designs. For instance, a BIB design with $v = m_1 m_2$ treatment combinations has little utility as a $m_1 \times m_2$ factorial, because both the main effects and interactions are confounded with block differences to the same extent. A regular or a semi-regular group divisible PBIB design with $v = m_1 m_2$ treatments is superior for a $m_1 \times m_2$ factorial as compared to a BIB design with the same number of treatments, because these designs do not only estimate one of the main effects free of block effects but also require a lesser number of replications.

Definition 3.1. A block design is said to have an orthogonal factorial structure if the adjusted sum of squares due to treatments in the block design can be partitioned orthogonally into a sum of squares corresponding to the main effects and a sum of squares corresponding to the interactions in a factorial experiment.

John and Smith (1972) and Cotter *et al.*(1973) have shown that if the matrix Ω of an incomplete block design has cyclic structure, then this design has factorial structure and therefore can be utilized as a factorial experiment. Since for most of designs the matrices M and Ω have the same pattern, PEB designs with matrices M having cyclic patterns can be used as factorial experiments.

Puri and Nigam (1976) have defined the balanced factorial experiment (BFE) as a generalization of the concept introduced by Shah (1960).

Definition 3.2. A factorial experiment with $v = m_1 m_2 \ldots m_h$ treatment combinations is called a BFE if the following conditions are satisfied:

(i) the i-th treatment combination is replicated r_i times,

(ii) the j-th block has k_j plots, j = 1,..., b,

(iii) estimates of contrasts belonging to different interactions are uncorrelated with each other,

(iv) a "complete balance" is achieved over each order of interaction.

According to Puri and Nigam (1976), a "complete balance" is achieved over an interaction (effect) if the relative loss of information for each contrast belonging to that interaction is the same. For instance, we say that complete balance is achieved over interaction $F_1^{c1} F_2^{c2} \ldots F_h^{ch}$, where $c_j = 0$ or 1, if all the $(m_1-1)^{c1}(m_2-1)^{c2}\ldots(m_h-1)^{ch}$ contrasts belonging to this interaction have the same relative loss of information.

Throughout this section the symbolic direct product $\theta_1 \otimes \theta_2 \otimes \ldots \otimes \theta_h$ will be used to order lexicographically the v treatment combinations. The matrix M of a BFE with $v = m_1 m_2 \ldots m_h$ treatment combinations has the spectral decomposition

$$M = \sum_{d=0}^{h} \sum_{c_1 + \ldots + c_h = d} \mu_{c_1 \ldots c_h} L_{c_1 \ldots c_h}, \qquad (10)$$

where $L_{c_1 \ldots c_h} = L_1^{c1} \otimes \ldots \otimes L_h^{ch}$,

$$L_j^{c_j} = \begin{cases} \{I_j - (1/N_j)1a_j'\} & \text{if } c_j = 1, \\ (1/N_j)1a_j' & \text{otherwise} \end{cases} \qquad (11)$$

and $a_j'1 = N_j$.

It clearly follows from the structure of M that a BFE is a PEB design with at most $2^h - 1$ efficiency classes. Many of these classes may coincide.

Puri and Nigam (1976) have introduced a property (A^*) as a generalization of a property (A) of Kurkjian and Zelen (1962, 1963).

Definition 3.3. An incomplete block design N with parameters

$$v = m_1 m_2 \ldots m_h,\ b,\ r = (a_1 \otimes \ldots \otimes a_h),\ k,\ \mu_{c_1 \ldots c_h}$$

is said to possess the property (A^*) if the matrix M has the spectral decomposition

$$M = \sum_{d=0}^{h} \sum_{c_1 + \ldots + c_h = d} g(c_1, \ldots, c_h) D_1^{c_1} \otimes \ldots \otimes D_h^{c_h}, \qquad (12)$$

where $g(c_1, \ldots, c_h)$ are constants depending on c_j and

$$D_j = \begin{cases} I_j & \text{if } c_j = 1, \\ (1/N_j)1a_j' & \text{otherwise.} \end{cases}$$

An incomplete block design possessing the property (A^*) is clearly a PEB design with at most $2^h - 1$ efficiency classes, many of which may coincide. The designs of randomized block (RB), orthogonal (O), BIB, EB, GD, L_2, as well as Kronecker product designs such as cubic, hypercubic, extended group divisible, rectangular and extended rectangular, and also the designs obtained from them by merging treatments suitably possess the property (A^*). It can easily be shown that the matrix M of a block design possessing property (A^*) can be expressed as (10) and therefore these block designs can be used as BFE. These designs have a very simple analysis. It may be remarked here that the incomplete block designs with property (A) form a subclass of the incomplete block designs of John and Smith (1972) having the factorial structure and it may be noted that the design with the factorial structure may not be factorially balanced in the sense of Puri and Nigam (1976).

4. PEB DESIGNS AS MATING DESIGNS

In this section we shall demonstrate the use of a PEB design in genetic experiments. Diallel crossings are becoming increasingly popular in plant breeding programs. Basically, there are two types of distinguishable diallel crossings. In diallel crossing of type I (Hinkelmann and Stern, 1960) or factorial mating (Cockerham, 1963), m male

lines and f female lines are crossed with each other, i.e. there are mf matings in a complete diallel crossing of this type. In another type of diallel crossing, called type II, n parents are crossed among themselves and there are $n(n-1)/2$ matings if the reciprocals and the parental inbreds are ignored.

In a complete diallel, the number of crosses increases with the increase in the number of parents, which creates the problem of resources and manageability. This leads to partial diallel crossings (PDC) where all the matings are not made. Here we shall deal with PDC of type I.

Suppose there are m male lines and f female lines. The PDC of type I are defined such that each male line is crossed with r_m females and each female line is crossed with r_f males. The total number of crosses is n, $n = mr_m = fr_f$, each cross having p offsprings. Hinkelmann (1966) proposed an explicit procedure of constructing PDC of type I from an incomplete binary block design. His procedure is as follow.

Consider any binary incomplete block design N with parameters v, b, r, k. Identify the male parents with treatments and female parents with blocks and include a cross (i x j) in the PDC if and only if the i-th treatment occurs in the j-th block of the design. We get PDC with parameters $m = v$, $f = b$, $r_m = r$, and $r_f = k$. If we identify males with blocks and females with treatments we get a PDC $m = b$, $f = v$, $r_m = k$, and $r_f = r$.

For the analysis of such PDC Hinkelmann (1966) has considered the usual two-way classification model, with unequal frequencies given below, under the assumption that v_i, w_j, s_{ij} and e_{ijk} are distributed with mean 0 and variances σ_m^2, σ_f^2, σ_s^2, and σ_e^2, respectively. The model has the form

$$y_{ijk} = \mu + v_i + w_j + r_k + s_{ij} + e_{ijk},$$
$$i = 1,\ldots, m, \quad j = 1,\ldots, f, \quad k = 1,\ldots, p,$$

where y_{ijk} is the yield of the (i x j)-th cross in the k-th replication if the (i x j)-th cross is included in PDC, μ is a general effect, v_i is the general combining ability (gca) effect of the i-th male line, w_j is the gca effect of the j-th female line, r_k is the k-th replication effect, s_{ij} is the specific combining ability of the (i x j)-th cross, and e_{ijk} is a random error.

Let N $(=n_{ij})$ denote the mxf incidence matrix of the PDC, where $n_{ij} = 1$ if the (i x j)-th cross is performed and 0 otherwise. For male and female lines the following matrices can be defined:

$$\Omega_m^{-1} = r_m I_m - (1/r_f)NN' + (1/m)1_m 1_m',$$

$$\Omega_e^{-1} = r_f I_f - (1/r_m)N'N + (1/f)1_f 1_f'.$$

In the PDC we are interested in estimating and comparing the gca effects of male lines as well as of female lines, therefore the knowledge

of the patterns of both the original design N and its dual design N^*(= N') is essential for the construction of PDC of type I. Hinkelmann (1966) has constructed PDC by using BIB designs and PBIB designs with two associate classes whose duals are of the same type. He has listed about 100 PDC for various values of m and f (≤ 30) and r_m and r_f (≤ 10). His list is not exhaustive and thus the PDC are not available for many parametric combinations, and, on the other hand, they need very often a large number of crosses.

We state a theorem due to Kageyama and Puri (1985a) on dual design without proof.

Theorem 4.1. The dual of a PEB design with parameters v, b, r, k, μ_i, ρ_i, L_i, i = 1,..., m, is a PED design having at most m + 1 efficiency classes with parameters

$$v^* = b,\ b^* = v,\ r^* = k,\ k^* = r,\ \mu_i^* = \mu_i,$$

$$L_i^* = (1/\mu_i)K^{-1}N'L_iR^{-1}N,\quad i = 1,\ldots, m.$$

Hence by the above theorem the duals of all incomplete block designs are known. Note that

(i) the dual of a symmetrical BIB is again a symmetrical BIB and the dual of a non-symmetrical BIB is a S-PEB design,

(ii) the dual of a EB design will be a EB or a S-PEB depending on whether v = b or b > v, respectively,

(iii) the dual of a S-PEB design is a EB or a S-PEB depending on whether ρ = b - 1, or $\rho \leq$ b - 1, respectively,

(iv) the dual of a PBIB(m) is a PEB with at most (m + 1) efficiency classes.

Since the duals of every BIB, PBIB and PEB designs are known, any available binary block design can be used as PDC. Even a binary PEB design with varying block sizes and varying replications can be used to construct PDC of type I, where the i-th male line is crossed with r_i, i = 1,..., m, females, and the j-th female line is crossed with k_j, j = 1,..., f, male lines. As μ_i's are known for almost all PBIB and known PEB designs, the experimenter will have no problem in making a selection of the most efficient PDC with desired parameters. Some optimal PDC based on BIB and S-PEB designs were presented by Puri *et al.*(1986).

5. PEB DESIGNS FOR BIOLOGICAL ASSAYS

Biological assays (bio-assays) constitute a field where the PEB designs can be used quite effectively. Unlike the situations in agricultural and industrial experiments, in bio-assays not all contrasts of treatment (dose) effects are of equal importance. For instance, in parallel line (PL) assays the contrasts of major importance are

preparation (L_p), combined regression (L_i) and parallelism (L_i'). The first two provide an estimate of relative potency while the third one is important for making validity tests. Consequently, it is desirable that the design for PL assays should estimate at least these three contrasts with maximum efficiency. The conventional incomplete block design like BIB and PBIB designs are not useful for PL assays as important contrasts get confounded with block differences in these designs. In the present section we shall discuss the desirable design patterns for PL assays which estimate important contrasts with maximum efficiency. For the design patterns for the slope ratio and multiline assays reference may be made to Gupta (1984) and for the quadratic parallel line assay reference may be made to Seshagiri (1974).

In a PL assays with $v = m_s + m_t$ preparations, where m_s and m_t are the number of doses for standard and test preparations, respectively, the (v - 1) degrees of freedom can be split into single degree of freedom contrasts L_p, L_h and L_h^*, h = 1,2,... , where L_h and L_h^* denote the sums and differences of the h-th power regression. It is useful to subdivide L_h and L_h^* further into odd and even numbered contrasts L_{2n+1}, L_{2n+2}, L_{2n+1}^* and L_{2n+2}^*, respectively.

If $m_s = m_t = m$, then the PL assay is called symmetrical (SPL), otherwise it is an asymmetrical (APL) assay. For the sake of simplicity we shall here discuss the desirable pattern of incomplete block designs for SPL assays. Let $s_1,\ldots, s_m$ and $t_1,\ldots, t_m$ denote the doses of standard and test preparations, respectively, on the logarithmic scale. Let the doses be arranged in the order

$$a_1,\ldots, a_p, a_{2p}, a_{2p-1},\ldots, a_{p+1},$$

if the number of doses is even, say m = 2p, , and in the order

$$a_1,\ldots, a_p, a_{2p+1}, a_{2p},\ldots, a_{p+1}$$

if the number of doses is odd, say m = 2p + 1, where $a_i = s_i$ or t_i.

The coefficients of L_p, L_{2n+1}, L_{2n+1}^*, L_{2n+2} and L_{2n+2}^* can be written as

$$l_p = (1 \quad -1)' \otimes 1_m,$$

$$l_{2n+1} = (1 \quad 1)' \otimes U_m, \qquad L_{2n+1}^* = (1 \quad -1)' \otimes U_m,$$

$$l_{2n+2} = (1 \quad 1)' \otimes V_m, \qquad l_{2n+2}^* = (1 \quad -1)' \otimes V_m,$$

respectively, where U_m and V_m are m x 1 vectors satisfying $U_m'1_m = V_m'1_m = 0$. Here U_m and V_m are orthogonal polynomials of degree 2n + 1 and 2n + 2, n = 0, 1,..., i.e. odd and even order polynomials. If m = 2p, then U_m and V_m can be written as

$$U_m = (1 \quad -1)' \otimes x_p, \qquad V_m = (1 \quad 1)' \otimes y_p,$$

where x_p and y_p are p x 1 vectors satisfying $y_p'1_p = 0$. If $m = 2p + 1$, then U_m and V_m can further be written as

$$U_m = [(1 \quad -1)' \otimes x_p', \ 0]', \qquad V_m = [(1 \quad 1)' \otimes y_p', \ 0]'$$

where $y_p'1_p \neq 0$.

For instance, if $m = 4$, then 7 mutually orthogonal contrasts for $m_1 = m_2 = 4$ PL assay can be written as

$$\begin{aligned}
l_p &= (1, \ 1, \ 1, \ 1, -1, -1, -1, -1)', \\
l_1 &= (3, \ 1, -3, -1, \ 3, \ 1, -3, -1)', \\
l_1^* &= (3, \ 1, -3, -1, -3, -1, \ 3, \ 1)', \\
l_2 &= (1, -1, \ 1, -1, \ 1, -1, \ 1, -1)', \\
l_2^* &= (1, -1, \ 1, -1, -1, \ 1, -1, \ 1)', \\
l_3 &= (1, -3, -1, \ 3, \ 1, -3, -1, \ 3)', \\
l_3^* &= (1, -3, -1, \ 3, -1, \ 3, \ 1, -3)'.
\end{aligned}$$

Here $U_m = (3, \ 1, -3, -1)' = (1 \quad -1)' \otimes (3 \quad 1)'$,
$V_m = (1, -1, \ 1, -1)' = (1 \quad 1)' \otimes (1 \quad -1)'$,
$x_1 = (3 \quad 1)'$ and $y_1 = (1 \quad -1)'$

for the first and second degree polynomials, respectively.

We state the following after Gupta *et al.* (1985).

Theorem 5.1. In a SPL assay with the matrix M having a pattern

$$M = \begin{bmatrix} A_m & B_m \\ B_m & A_m \end{bmatrix}$$

the contrasts L_p, L_{2n+1} and L_{2n+1}^* will be estimated with full efficiency if the m x m matrices A and B have the pattern

$$A_m = J_2 \otimes A_p \quad \text{and} \quad B_m = J_2 \otimes B_p$$

in the case of even m ($m = 2p$), and

$$A_m = \begin{bmatrix} J_2 \otimes A_p & d_1 \\ d_1' & a_1 \end{bmatrix} \quad \text{and} \quad B_m = \begin{bmatrix} J_2 \otimes B_p & d_2 \\ d_2' & a_2 \end{bmatrix}$$

in the case of odd m ($m = 2p + 1$), where A_p and B_p are p x p matrices, d_1 and d_2 are 2p x 1 vectors of non-negative real numbers and a_1, a_2 are positive constants and functions of design parameters. Further, if $AU_m = \alpha U_m$ and $BV_m = \beta V_m$ then L_{2n+2} and L_{2n+2}^* are estimated with losses $\alpha + \beta$ and $\alpha - \beta$, respectively. In this case all the bio-assay contrasts are basic contrasts and design is a PEB with at most three efficiency classes. Particularly, if $\alpha = \beta$ (or $\alpha = -\beta$) then L_{2n+2}^* (or L_{2n+2}^*) becomes free from block effects and the design is a S-PEB.

Example 5.1. Let $m = 2p$, then the design with blocks

$$(s_i, \ s_{2p-i+1}, \ t_j, \ t_{2p-j+1}), \quad i = 1, \ldots, p,$$

is a S-PEB design with parameters $v = 2m$, $b = p^2$, $r = p$, $k = 4$, $\mu = 1/2$ and with the matrix M of the form

$$M = (1/rk) \begin{bmatrix} rJ_2 \otimes I_p & J_2 \otimes J_p \\ J_2 \otimes J_p & rJ_2 \otimes I_p \end{bmatrix}.$$

Many methods of construction of an incomplete block design for PL assays having a pattern of theorem 5.1 have been presented by Nigam and Boopathy (1985) and Gupta *et al.* (1985). PEB designs estimating only L_p, L_1 and L_1^* with full efficiency may not have the pattern of theorem 5.1. These designs were constructed by Kyiwin and Dey (1980) by trial and error. Systematic methods of construction of such designs for APL assays have been discussed by Puri and Gupta (1985a, 1985b, 1986) and Puri *et al.*(1985).

6. PEB DESIGNS FOR COMPARATIVE TRIALS

In many comparative experiments there may be a treatment (treatments), usually the control (controls), which may be logically at a different footing from the rest of treatments and the experimenter may like to include it (or them) at least in every block. For instance, in a factorial experiment, a control or a treatment combination with doses higher than the experimental ones, or a fresh treatment is included. Designs such as reinforced (Das, 1958), supplemented balanced (Pearce, 1960), orthogonally supplemented block (Caliński, 1971), supplemented block (Caliński and Ceranka, 1974, and Puri *et al.*, 1977) and augmented factorial (Puri *et al.*, 1984) have been developed for these circumstances. The basic approach in all such designs is to supplement any standard design with a control (or controls). In these designs interest is centered on contrasts involving control and other treatments. Other contrasts involving only basic treatments or only controls are of less or of no importance.

Puri and Kageyama (1985) discussed the most general case of supplemented PEB designs and most of the work referred to the above emerged, as particular cases, of their work. However, we shall here present a particular case of these designs because of their simplicity and their importance.

Let N be the incidence matrix of an equi-block sized PEB design with parameters v, b, r, k, μ_i, ρ_i, L_i, $i = 1,\ldots, m$. Add one or more (say s) supplementary treatments to each block of the design N. Let the i-th supplementary treatment be added a_i times to every block of the basic design N. Then the incidence matrix of the resulting design is

$$N^* = \begin{bmatrix} N \\ a \otimes 1_b' \end{bmatrix}, \quad \text{where} \quad a = (a_1, \ldots, a_s)'.$$

It can easily be shown that the matrix M_0^* corresponding to N^* has the pattern

$$M_0^* = \begin{bmatrix} \{k/(k + k_0)\}M_0 & O \\ O & O \end{bmatrix} = \sum_{i=1}^{m} \mu_i^* L_i^*,$$

where $\mu_i^* = k\mu_i/(k + k_0)$, $L_i^* = \text{diag}(L_i, O)$ and $k_0 = 1_s'a$. That is, N^* is a PEB design with at most $m + 1$ efficiency classes with idempotent matrices L_i^* and eigenvalues μ_i^*.

Since a BIB design with parameters v, b, r, k, λ is a trivial PEB design with parameters v, b, r, k, $\mu = (r - \lambda)/rk$, $\rho = v - 1$, and $L = (I - \{1/v\}11')$, the supplemented BIB design obtained through the above procedure is a S-PEB design. If $a_i = 1$ for all i, then we get, as a particular case, the design of type A given by Corsten (1962). If s = 1, then the designs fall in the category of supplemented balanced designs of Pearce (1960).

If s_{ij}, $j = 1,\ldots, \rho_i$, denote the ρ_i contrasts of the i-th class which were estimated with relative loss μ_i, then these contrasts are estimated with an efficiency factor $1 - \mu_i^*$ in the new design which is greater than $1 - \mu_i$, the efficiency factor associated with these contrasts in the original design. Any contrast between the supplementary treatments and an intergroup contrast between the original treatments versus supplementary treatments is estimated with the efficiency factor 1.

In the supplemented designs obtained above, every supplementary treatment has to be included at least once in every block and therefore the number of replications required for supplementary treatments becomes large. This could be inconvenient in many experimental situations, especially when a large number of trials are to be conducted over a number of years and locations. This is all the more so when high costs of control treatments are involved. Utilizing the availability of α-resolvable PEB designs and more than one control, Puri and Kageyama (1985) have given supplementary designs for comparative trials where the supplementary treatments need fewer replications and at the same time every supplementary treatment occurs an equal number of times with every treatment of the basic design to ensure that the contrasts involving supplementary treatments and control are estimated with maximum efficiency.

An equi-replicated and equi-block sized PEB design is called α-resolvable if the blocks can be partitioned into t sets of β blocks, so that the set consisting of β blocks contains every treatment exactly α times. For an α-resolvable PEB design we have $b = t\beta$, $r = \alpha t$, $v\alpha = k\beta$.

We obtain N^* by adding s supplementary treatments so that the first supplementary treatment is added to all the $a_1\beta$ blocks of a_1 sets, the second supplementary treatment to $a_2\beta$ blocks of a_2 sets and so on so, that $\sum_{i=1}^{s} a_i = t$. The incidence matrix N^* of the resulting design is

$$N^* = \begin{bmatrix} N \\ \text{diag}(1'_{a_1\beta}, \ldots, 1'_{a_s\beta}) \end{bmatrix}$$

and M_0 has the pattern

$$M_0^* = \begin{bmatrix} \{k/(k+1)\}M_0 & O \\ O & \{1/(k+1)\}I_t - (1/t)1a' \end{bmatrix},$$

where $a = (a_1, \ldots, a_s)'$. Hence N^* is a PEB design with at most $m + 2$ efficiency classes with idempotent matrices $L_i^* = \text{diag}(L_i, O)$, $i = 1, \ldots, m$, $L_{m+1}^* = \text{diag}(O, I_t - \{1/t\}1a')$, $L_{m+2}^* = I_{v+s} - \sum_{i=1}^{m+1} L_i^*$, $L_0^* = \{1/b(r+1)\}1r^{*\prime}$ and losses of information $\mu_i^* = k\mu_i/(k+1)$, $\mu_{m+1}^* = 1/(k+1)$ and $\mu_{m+2}^* = 0$.

In the resultant design the contrasts between the basic treatments are estimated with the relative loss μ_i^*, the contrasts between supplementary treatments with the efficiency factor $k/(k+1)$, and the intergroup contrast between two sets of treatments with full efficiency.

The generalization of this result in many directions has been dealt with by Puri and Kageyama (1985). They have also demonstrated the utility of $(\alpha_1, \ldots, \alpha_s)$ resolvable PEB designs to construct supplementary designs.

7. PEB DESIGNS FOR SCREENING TRIALS

In plant breeding experiments with new strains or in plant protection experiments with new pesticides, herbicides, soil fumigants etc. a major problem is the screening of new strains of chemicals. In these experiments we generally come across the situations where limited material just sufficient for making one or two observations for new strains or chemicals is available. For such cases the augmented designs introduced by Federer (1961) are extensively used in practice. These designs are obtained by augmenting any standard design in check (standard) treatments with new strains (treatments) requiring a single observation. Federer (1961) considered the most general case. His designs are available in varying block sizes also. He has given the analysis by the method of fitting constants which is quite cumbersome and time consuming. We shall now discuss some particular cases of augmented designs which are generally used in practice and have simple analysis being PEB designs.

Case I. Check treatments in RBD.

Let N be the incidence matrix of RBD with v treatments and with r replications. Let there be $s = ab$ new strains (treatments) available requiring a single observation. Obtain N^* by augmenting each block with a new treatments, so that each new treatment has a single replication. The incidence matrix of the resultant design is

$$N^* = \begin{bmatrix} 11' \\ I_b \otimes 1 \end{bmatrix}$$

and the corresponding matrix M_0^* has the pattern

$$M_0^* = \{1/(v + \alpha)\}\text{diag}(0, [I_b - \{1/b\}J_b] \otimes J_\alpha).$$

Hence N^* is a S-PEB design with parameters $v^* = v + s$, $b^* = b$, $r^* = (b1_v', 1_s')'$, $k^* = v+\alpha$, $\mu^* = \alpha/(v+\alpha)$, $L^* = \text{diag}(0, \{1/\alpha\}[I_b - \{1/b\}J_b] \otimes J_\alpha)$, $\rho^* = b - 1$.

In this series only the contrasts between the new treatments belonging to different blocks are confounded with block differences and are estimated with efficiency factor $v/(v + \alpha)$, whereas all other contrasts are estimated with full efficiency.

Case II. Check treatments are in the linked block design.

A design is called a linked block design if every pair of blocks have the same number of treatments in common. Let N be the incidence matrix of a BIB design. Then N' is the incidence matrix of a linked block design. If we start with a linked block design as a basic design with b standard treatments and add $s = v\alpha$ new treatments, as explained in case I, then the resultant design is a S-PEB with parameters $v^* = b + \alpha v$, $b^* = v$, $r^* = (k1', \alpha v1')'$, $k^* = r + \alpha$, $\mu^* = (r + \alpha k - \lambda)/k(r + \alpha)$, L^*, $\rho = v - 1$.

Case III. Check treatments in a BIB design.

Let N be the incidence matrix of a BIB design, with v, b, r, k, λ, then the augmented design obtained, as in the first case, by adding $s = \alpha b$ new treatments is a PEB design with three efficiency classes and with losses $\{r(\alpha + 1) - \lambda\}/\{r(k + \alpha)\}$, $\alpha/(k + \alpha)$ and 0, respectively. The corresponding idempotent matrices can be easily obtained on the line of Puri and Kageyama (1985).

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